Degree Spectra of Structures with Low Scott Rank

Hongyu Zhu

University of Wisconsin-Madison

Joint work with Uri Andrews, David Gonzalez, and Joseph S. Miller.

Table of Contents

Background: Infinitary Logic

We extend first-order logic to the infinitary logic $\mathcal{L}_{\omega_1\omega}$ by adding countable conjunctions and disjunctions, while quantification remains limited to finitely many variables. Countable conjunctions and disjunctions will be denoted by special symbols: $\bigwedge\!\!\!\!\bigwedge, \bigvee\!\!\!\!\bigvee$.

The following hierarchy is analogous to the arithmetical hierarchy, and we treat countable conjunctions/disjunctions as universal/existential quantifiers.

Definition

- $\Sigma_0^{\mathrm{in}} = \Pi_0^{\mathrm{in}}$ consists of all *finitary* quantifier-free formulas.
- For a countable $\alpha > 0$ and formulas φ, ψ, say φ is $\Sigma_\alpha^{\mathrm{in}}$ and ψ is Π_α^{in} if

$$\varphi \equiv \bigvee\!\!\!\!\bigvee_{i \in I} \exists \overline{x} \chi_i(\overline{x}), \psi \equiv \bigwedge\!\!\!\!\bigwedge_{i \in I} \forall \overline{x} \theta_i(\overline{x}),$$

where each χ_i is Π_β^{in} and each θ_i is $\Sigma_\beta^{\mathrm{in}}$ for some $\beta < \alpha$.

Background: Scott Rank

Theorem (Scott Isomorphism Theorem)

Every countable structure $\mathcal{M}$ has an $\mathcal{L}_{\omega_1\omega}$ Scott sentence, i.e. a sentence φ which characterizes $\mathcal{M}$ up to isomorphism among countable structures.

The Scott sentence allows us to define a measure of complexity for a countable model, using a countable ordinal known as the *Scott rank*. Various similar notions go by the same name, and we use the following definition:

Theorem (Montalbán)

Given a countable structure $\mathcal{A}$ and a countable α, the following are equivalent:

- *$\mathcal{A}$ has a $\Pi^{\mathrm{in}}_{\alpha+1}$ Scott sentence.*
- *Every automorphism orbit of $\mathcal{A}$ is $\Sigma^{\mathrm{in}}_{\alpha}$-definable.*
- *$\mathcal{A}$ is uniformly $\mathbf{\Delta}^0_\alpha$-categorical. (etc.)*

The least such α is called the Scott rank of $\mathcal{A}$.

Background: Scott Rank Examples

Example

- $(\mathbb{N}, +, \cdot, 0, 1)$ has Scott rank 1, i.e. has a Π_2^{in} Scott sentence.

$$\forall x \bigvee_{n \in \omega} x = n.$$

- $(\mathbb{Q}, <)$ also has Scott rank 1; In fact it has a $\forall\exists$ Scott sentence.
- $(\omega^\alpha, <)$ has Scott rank 2α.
- (Montalbán, Rossegger) Nonstandard models of PA have Scott rank at least ω.

Background: Degree Spectrum

Another way of looking at the complexity of a countable structure is by investigating how much computational power is required to compute it.
From now on, assume the language is computable.

Definition

The *degree spectrum* $\mathrm{Spec}(\mathcal{M})$ of a countable structure $\mathcal{M}$ is the set of Turing degrees computing a copy of $\mathcal{M}$ (i.e. the atomic diagram of some $\mathcal{N}$ such that $\mathcal{N} \cong \mathcal{M}$).

Theorem (Knight)

If $\mathbf{d} \in \mathrm{Spec}(\mathcal{M})$, then there is a copy of $\mathcal{M}$ with Turing degree $\mathbf{d}$, unless $\mathcal{M}$ is automorphically trivial (in which case all copies of $\mathcal{M}$ have the same Turing degree).

From now on, we work with only automorphically non-trivial structures.

Background: Degree Spectrum Examples

Example

- The largest possible spectrum contains all Turing degrees, and comes from computable structures.
  These include: $(\mathbb{N}, +, \cdot, 0, 1)$; $(\mathbb{Q}, <)$; $(\omega^\alpha, <)$ for computable α, etc.
- (Tennenbaum) Nonstandard models of PA have no computable models.
- The second largest spectrum is the *Slaman-Wehner spectrum*, consisting of all the non-computable degrees.
- Every upper cone $\mathcal{C}_{\mathbf{d}} = \{\mathbf{x} \in \mathcal{D}_T \mid \mathbf{x} \geq \mathbf{d}\}$ is a degree spectrum.
- The last two are examples of a *family spectrum*, i.e. a spectrum consisting of all Turing degrees that can enumerate a fixed family of sets.

Family spectra are relatively simple, and can be realized by structures with Scott rank 1. Are there spectra that can't be realized by a structure with "low" Scott rank?

Background: Jump Inversions

Recall that every cone $\mathcal{C}_{\mathbf{d}} = \{\mathbf{x} \mid \mathbf{x} \geq \mathbf{d}\}$ is a degree spectrum. We can jump invert a spectrum to get a new one:

Definition

$\mathcal{C}_{\mathbf{d}}^{(-\alpha)} = \{\mathbf{x} \mid \mathbf{x}^{(\alpha)} \geq \mathbf{d}\}$.

The α-th jump inversion of a spectrum is still a spectrum (for non-limit α). However, the Scott rank of the underlying structure might go up.

Remark

In general, if $\mathcal{S}$ is a spectrum of a structure with Scott rank α, then $\mathcal{S}^{(-\beta)}$ is a spectrum of a structure with Scott rank (about) $\beta + \alpha$.

So a sufficiently high jump inversion might force us away from structures with "low" Scott rank. We start with "low" being 1.

The Existentially Atomic Case: Definition

Definition

A structure is *existentially atomic* if it has Scott rank 1.

Letting $\alpha = 1$ in the definition of Scott rank:

Remark

$\mathcal{A}$ being existentially atomic is equivalent to saying any of the following:

- $\mathcal{A}$ has a Π_2^{in} Scott sentence.
- Every automorphism orbit of $\mathcal{A}$ is existentially definable.

The second condition means that every $\overline{a} \in \mathcal{A}$ is isolated by a *finitary* existential formula $\varphi(\overline{x})$; And any two tuples satisfying φ are automorphic.

The Existentially Atomic Case: Building a Model

Recall: If $\mathcal{A}$ is existentially atomic, then every $\overline{a} \in \mathcal{A}$ is isolated by a *finitary* existential formula $\varphi(\overline{x})$; And any two tuples satisfying φ are automorphic. This suggests building $\mathcal{A}$ from the following information:

Definition

- $\mathcal{A}^{\text{fin}}$ is the set of all finite substructures of $\mathcal{A}$.
- $\mathcal{SF}_{\mathcal{A}} = (\mathcal{I}_{\mathcal{A}}, \sqsubseteq)$, where $\mathcal{I}_{\mathcal{M}}$ is the set of all $\exists_1$ formulas that isolate an automorphism orbit in $\mathcal{M}$, and $\sqsubseteq$ represent the relation on $\mathcal{I}_{\mathcal{M}}$ such that:

$$\varphi \sqsubseteq \psi \iff \text{Some realization of } \varphi \text{ is a subtuple of some realization of } \psi.$$

Remark

- We can find these in a Π^0_2 way from any presentation of $\mathcal{A}$.
- They essentially allow us to compute a Π^{in}_2 Scott sentence.

The Existentially Atomic Case: Building a Model

Theorem

Suppose $\mathcal{A}$ is existentially atomic, and $\mathbf{d}$ enumerates $\mathcal{A}^{\text{fin}}$ and $\mathcal{SF}_{\mathcal{A}}$. Then $\mathbf{d} \in \text{Spec}(\mathcal{A})$.

Proof Sketch.

Do an effective Fraïssé construction with additional requirements: Build $\mathcal{B} = \bigcup_s \mathcal{B}_s$ where each $\mathcal{B}_s \in \mathcal{A}^{\text{fin}}$. Start with $\mathcal{B}_0 = \varnothing$. Complete the following tasks:

- Extend to a larger $\mathcal{B}_{s+1} \in \mathcal{A}^{\text{fin}}$ (consistent with our commitments).
- Make sure all tuples have isolating formulas consistent with $\mathcal{SF}_{\mathcal{A}}$.
- Add witnesses to the isolating formulas (which are existential).
- "Homogenize" elements satisfying the same isolating formula.

This is more or less a relativization of arguments in III.6 of Montalbán, which focuses more on *effectively* existentially atomic structures. □

The Existentially Atomic Case: Minimal Pairs

At this point it may seem as if we achieved nothing: Assuming $\mathbf{b}$ computes a copy of $\mathcal{A}$, we could find a copy of $\mathcal{A}$ that is $\Pi^0_2(\mathbf{b})$, in particular computable from $\mathbf{b}''$. Not an impressive feat, since $\mathrm{Spec}(\mathcal{A})$ is upwards closed.

However, it is crucial that $X_\mathcal{A} = (\mathcal{A}^{\mathrm{fin}}, \mathcal{SF}_\mathcal{A})$ is *independent of the presentation of* $\mathcal{A}$. So for example, if $\mathrm{Spec}(\mathcal{A})$ contains $\mathbf{a}, \mathbf{b}$ with $\Pi^0_2(\mathbf{a}) \cap \Pi^0_2(\mathbf{b}) \subseteq \Pi^0_2(\mathbf{d})$, then $X_\mathcal{A} \in \Pi^0_2(\mathbf{d})$, so $\mathbf{d}'' \in \mathrm{Spec}(\mathcal{A})$. Now, $\mathbf{d}''$ may well be incomparable with $\mathbf{a}$ and $\mathbf{b}$, so we've gained something new.

Corollary

If $\mathrm{SR}(\mathcal{A}) = 1$ *and* $\mathbf{a}, \mathbf{b} \in \mathrm{Spec}(\mathcal{A}), \mathbf{d} \in \mathcal{D}_T$ *are such that* $\Sigma^0_2(\mathbf{a}) \cap \Sigma^0_2(\mathbf{b}) \subseteq \Sigma^0_2(\mathbf{d})$, *then* $\mathbf{d}'' \in \mathrm{Spec}(\mathcal{A})$.

The Existentially Atomic Case: Minimal Pairs

The condition in the previous corollary motivates the following definition.

Definition

$\mathbf{a}, \mathbf{b}$ forms a Σ^0_n-*minimal pair relative to* $\mathbf{d}$ if $\Sigma^0_n(\mathbf{a} \oplus \mathbf{d}) \cap \Sigma^0_n(\mathbf{b} \oplus \mathbf{d}) = \Sigma^0_n(\mathbf{d})$.

Remark

- Σ^0_n-minimal pair$=$ Π^0_n-minimal pair.
- If $\mathbf{d} = \mathbf{0}$, then we just say Σ^0_n-minimal pair.
- If $\mathbf{a}', \mathbf{b}'$ forms a Σ^0_1-minimal pair relative to $\mathbf{d}'$, then

$$\Sigma^0_2(\mathbf{a}) \cap \Sigma^0_2(\mathbf{b}) = \Sigma^0_1(\mathbf{a}') \cap \Sigma^0_1(\mathbf{b}') \subseteq \Sigma^0_1(\mathbf{d}') = \Sigma^0_2(\mathbf{d}).$$

  In particular, $\mathbf{a}, \mathbf{b}$ forms a Σ^0_2-minimal pair if and only if $\mathbf{a}', \mathbf{b}'$ forms a Σ^0_1-minimal pair relative to $\mathbf{0}'$.

Now it suffices to find (relativized) Σ^0_1-minimal pairs, which are relatively well-understood. The following is the relativized version of a lemma of Andrews and J. S. Miller.

Lemma

Given a structure $\mathcal{A}$ and $\mathbf{d} \in \mathcal{D}_T$, the following are equivalent:

1. *$\mathcal{A}$ has the ceep relative to $\mathbf{d}$, i.e. all $\exists$-types of $\mathcal{A}$ are c.e. in $\mathbf{d}$.*
2. *$\mathrm{Spec}(\mathcal{A})$ contains a Σ^0_1-minimal pair relative to $\mathbf{d}$.*
3. *$\mathrm{Spec}(\mathcal{A})_{\geq \mathbf{d}}$ contains a Σ^0_1-minimal pair relative to $\mathbf{d}$.*

Note that the equivalence of the last two conditions uses only the fact that $\mathrm{Spec}(\mathcal{A})$ is upwards closed.

Now consider $\mathcal{C}_{\mathbf{z}}^{(-2)}$ with $\mathbf{z} = \mathbf{0}^{(5)}$. It does not contain $\mathbf{0}''$, so it remains to show $(\mathcal{C}_{\mathbf{z}}^{(-2)})' = (\mathcal{C}_{\mathbf{z}}^{(-1)})_{\geq \mathbf{0}'}$ has a Σ_1^0-minimal pair relative to $\mathbf{0}'$.

Lemma

Given $\mathbf{x}$, $\mathbf{y}$: $(\mathcal{C}_{\mathbf{x}}^{(-1)})_{\geq \mathbf{y}} = \{\mathbf{d} \geq \mathbf{y} \mid \mathbf{d}' \geq \mathbf{x}\}$ *contains a* Σ_1^0*-minimal pair relative to* $\mathbf{y}$.

Proof.

By the previous lemma, it suffices to show $\mathcal{C}_{\mathbf{x}}^{(-1)}$ contains some $\mathrm{Spec}(\mathcal{A})$ where $\mathcal{A}$ has the ceep relative to $\mathbf{y}$. Richter showed that trees, *in the language* $\leq$ *of partial orders*, have the ceep. Now code some $X \in \mathbf{x}$ in the jump of a tree. □

As a result:

Theorem

$\{\mathbf{d} \mid \mathbf{d}'' \geq \mathbf{0}^{(5)}\}$ *is a degree spectrum, but not one of a structure with Scott rank 1.*

Proof.

- $\{\mathbf{d} \mid \mathbf{d}'' \geq \mathbf{0}^{(5)}\}$ is a jump inversion of a cone, thus a degree spectrum.
- If $\mathcal{S}$ is an $\exists$-atomic spectrum, then $\mathcal{S}$ having a Σ^0_2-minimal pair implies $\mathbf{0}'' \in \mathcal{S}$.
- $\{\mathbf{d} \mid \mathbf{d}'' \geq \mathbf{0}^{(5)}\}$ has a Σ^0_2-minimal pair (since its jump has a Σ^0_1-minimal pair relative to $\mathbf{0}'$).
- But $\mathbf{0}'' \notin \{\mathbf{d} \mid \mathbf{d}'' \geq \mathbf{0}^{(5)}\}$, so $\{\mathbf{d} \mid \mathbf{d}'' \geq \mathbf{0}^{(5)}\}$ is not an $\exists$-atomic spectrum.

□

The General Case: Reduction

We would like to reduce to the existentially atomic case. For this we use the (lightface) α-jump of a structure, which essentially adds predicates for all computable Σ_α formulas to $\mathcal{A}$ (uniformly), and has the following properties.

Fact (See for example Montalbán)

Given a structure $\mathcal{A}$ and a computable α, there exists $\mathcal{A}^{(\alpha)}$ such that:

- $\mathrm{Spec}(\mathcal{A}^{(\alpha)}) = \mathrm{Spec}(\mathcal{A})^{(\alpha)}$;
- *If $\mathcal{A}$ has a computable $\Pi_{\alpha+2}$ Scott sentence, then $\mathcal{A}^{(\alpha)}$ has a computable Π_2 Scott sentence.*

However, in general a structure's Scott sentence (even if Π_2^{in}) may be very far from being computable.

The General Case: Reduction

But if we assume some computability to start with:

Proposition

Let $\alpha, \beta < \omega_1^{CK}$. If $\mathrm{SR}(\mathcal{A}) \leq \alpha$ and $\mathcal{A}$ is $\mathbf{0}^{(\beta)}$-computable, then $\mathrm{SR}(\mathcal{A}^{(\beta+2\alpha)}) = 1$. (In fact, $\mathcal{A}^{(\beta+2\alpha)}$ has a computable Π_2 Scott sentence.)

Proof.

Using the back-and-forth relations, one can write down a $\mathbf{0}^{(\beta)}$-computable $\Pi_{2\alpha+2}$ Scott sentence for $\mathcal{A}$ (relativizing the usual proof that computable structures with Π_α^{in} Scott sentences have computable $\Pi_{2\alpha}$ Scott sentences). Thus, $\mathcal{A}$ has a computable $\Pi_{\beta+2\alpha+2}$ Scott sentence, so $\mathcal{A}^{(\beta+2\alpha)}$ has a computable Π_2 Scott sentence. □

Now we aim to replicate the remaining arguments with the following outline: Fix $\mathcal{A}$ with Scott rank at most α. Suppose towards a contradiction that $\mathrm{Spec}(\mathcal{A}) = \mathcal{C}_{\mathbf{0}^{(\nu)}}^{(-\mu)} = \left\{\mathbf{d} \mid \mathbf{d}^{(\mu)} \geq \mathbf{0}^{(\nu)}\right\}$, then:

- Find β such that $\mathbf{0}^{(\beta)} \in \mathrm{Spec}(\mathcal{A})$.
- Find a suitable γ, which makes $\mathcal{B} = \mathcal{A}^{(\gamma)}$ $\exists$-atomic (we can use $\gamma = \beta + 2\alpha$).
- $\mathrm{Spec}(\mathcal{B})' = (\mathrm{Spec}(\mathcal{A}))^{(\gamma+1)}$ has a Σ_1^0-minimal pair relative to $\mathbf{0}^{(\gamma+1)}$.
- Conclude that $\mathbf{0}^{(\gamma+2)} \in \mathrm{Spec}(\mathcal{B})$.
- Obtain a contradiction using $\mathrm{Spec}(\mathcal{B}) = \left\{\mathbf{d}^{(\gamma)} \mid \mathbf{d}^{(\mu)} \geq \mathbf{0}^{(\nu)}\right\}$.

The General Case: Minimal Pairs

Recall:

Lemma

Given $\mathbf{x}, \mathbf{y}$*:* $\{\mathbf{d} \geq \mathbf{y} \mid \mathbf{d}' \geq \mathbf{x}\}$ *contains a* Σ^0_1*-minimal pair relative to* $\mathbf{y}$*.*

Want: $\mathrm{Spec}(\mathcal{B})' = (\mathrm{Spec}(\mathcal{A}))^{(\gamma+1)}$ has a Σ^0_1-minimal pair relative to $\mathbf{0}^{(\gamma+1)}$.

Using $\mathrm{Spec}(\mathcal{A}) = \left\{\mathbf{d} \mid \mathbf{d}^{(\mu)} \geq \mathbf{0}^{(\nu)}\right\}$:

$$(\mathrm{Spec}(\mathcal{A}))^{(\gamma+1)} = \left\{\mathbf{d}^{(\gamma+1)} \mid \mathbf{d}^{(\mu)} \geq \mathbf{0}^{(\nu)}\right\}.$$

To make it fit the lemma, take $\mu = \gamma + 2$ and use $(\gamma + 1)$-jump inversion:

$$(\mathrm{Spec}(\mathcal{A}))^{(\gamma+1)} = \left\{\mathbf{d}^{(\gamma+1)} \mid \mathbf{d}^{(\gamma+2)} \geq \mathbf{0}^{(\nu)}\right\} = \left\{\mathbf{a} \geq \mathbf{0}^{(\gamma+1)} \mid \mathbf{a}' \geq \mathbf{0}^{(\nu)}\right\}$$

The General Case: Ordinal Arithmetic

Further computation gives:

$$\mathrm{Spec}(\mathcal{A}) = \left\{\mathbf{d} \mid \mathbf{d}^{(\beta+2\alpha+2)} \geq \mathbf{0}^{(\beta+2\alpha+5)}\right\}.$$

Since $\mathbf{0}^{(\beta)} \in \mathrm{Spec}(\mathcal{A})$, we want β to be least such that $\beta + \beta + 2\alpha > \beta + 2\alpha + 2$.

- If $\alpha = n$ is finite: Take $\beta = 3$.
- If α is infinite: Take β to be the least λ such that $\lambda + \alpha > \alpha$.
  - Equivalently, the greatest additively closed ordinal not exceeding α;
  - Equivalently, the leading term in the Cantor normal form of α.
  - $\lambda \leq \alpha$.

The General Case

Theorem

Let α be computable. Then the following is a degree spectrum, but not one of a structure with Scott rank $\leq \alpha$.

- *For finite $\alpha = n$:* $\{\mathbf{d} \mid \mathbf{d}^{(2n+5)} \geq \mathbf{0}^{(2n+8)}\}$.
- *For infinite α:* $\{\mathbf{d} \mid \mathbf{d}^{(\lambda+2\alpha+2)} \geq \mathbf{0}^{(\lambda+2\alpha+5)}\}$, *where λ is least such that $\lambda + \alpha > \alpha$.*

Remark

The current bounds probably need sharpening. For example, it is not optimal when $n = 1$. Work is in progress to find the best bounds.

Thank you for listening!