(Non-)
Absoluteness
of ℕ

Hongyu
Zhu

Survey

Defining ℕ

Is There a
Problem?

What to
Do?

Bibliography

Are We Talking about the Same Natural Number(s)?
Thinking Deeply about Simple Things

Hongyu Zhu

University of Wisconsin-Madison

The 23rd Graduate Student Conference in Logic
April 15, 2023

Table of Contents

(Non-)
Absoluteness
of ℕ

Hongyu
Zhu

Survey
Defining ℕ
Is There a
Problem?
What to
Do?
Bibliography

Survey

Which Notions Are Absolute (if any)?

(Non-)
Absoluteness
of ℕ

Hongyu
Zhu

Survey

Defining ℕ

Is There a
Problem?

What to
Do?

Bibliography

Feel free to interpret "absoluteness" in your own way.

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- Set theoretic truths

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(Non-)
Absoluteness
of ℕ

Hongyu
Zhu

Survey

Defining ℕ

Is There a
Problem?

What to
Do?

Bibliography

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(Non-)
Absoluteness
of ℕ

Hongyu
Zhu

Survey

Defining ℕ

Is There a
Problem?

What to
Do?

Bibliography

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- Finiteness

(Non-)
Absoluteness
of ℕ

Hongyu
Zhu

Survey

Defining ℕ

Is There a
Problem?

What to
Do?

Bibliography

Popular Beliefs

- Absolutely absolute (ZFC, large cardinals, . . .)
- First-order truths are absolute, higher-order ones are not
- Multiverse ("possible worlds")
- . . .

Defining ℕ

$\mathbb{N} = \{0, 1, 2, \dots\}$

The natural numbers are $0, 1, 2$, and so on.

Intuitive Definition

$\mathbb{N} = \{0, 1, 2, \dots\}$

The natural numbers are $0, 1, 2,$ **and so on**.

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The natural numbers are $0, 1, 2$, and so on.

The least system containing $\{0, 1, 2\}$, closed under certain operations...?

Rigorous Definition

- Peano Arithmetic

Rigorous Definition

- Peano Arithmetic
- Second-Order PA

Rigorous Definition

- Peano Arithmetic
- Second-Order PA
- Axiom of Infinity

Are we using a "Church-Turing thesis"?

Is There a Problem?

- Intuitive Definition: Loophole? Circular? Philosophical only? Reduces to rigorous definition?

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- First-order PA: Non-standard models exist.

- Intuitive Definition: Loophole? Circular? Philosophical only? Reduces to rigorous definition?
- First-order PA: Non-standard models exist.
- Second-order PA: Categorical.
- ω: Unique by definition.

- Unique, simple definition.
- Clear-cut: e.g. Tennenbaum's Theorem.

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. . . Or so it seems.

- Delegating to underlying set theory.

- Unique, simple definition.
- Clear-cut: e.g. Tennenbaum's Theorem.

. . . Or so it seems.

- Delegating to underlying set theory.
- In fact, we can build a set S and 3 universes V_0, V_1, V_2, such that: S is finite in V_0, countably infinite in V_1, and uncountable in V_2. (with not-so-complicated techniques)

Hamkins: list of axioms for a collection of set-theoretic universes
("multiverse").

> **_Well-foundedness Mirage._** _Every universe V is ill-founded from_
> _the perspective of another, better universe._"

Hamkins: list of axioms for a collection of set-theoretic universes
("multiverse").

> ***Well-foundedness Mirage.*** *Every universe V is ill-founded from
> the perspective of another, better universe.*

In particular, every (universe's) ω is non-standard when viewed by another one.

Hamkins: list of axioms for a collection of set-theoretic universes ("multiverse").

> **Well-foundedness Mirage.** *Every universe V is ill-founded from the perspective of another, better universe.*

In particular, every (universe's) ω is non-standard when viewed by another one.

Mathematical support: The collection of all countable, computably saturated models of ZFC serves as a "toy multiverse."

Summary

(Non-)
Absoluteness
of ℕ

Hongyu
Zhu

Survey

Defining ℕ

Is There a
Problem?

What to
Do?

Bibliography

There is mathematical evidence showing that finiteness is not absolute.

Summary

(Non-) Absoluteness of ℕ

Hongyu Zhu

Survey

Defining ℕ

Is There a Problem?

What to Do?

Bibliography

There is **mathematical evidence** showing that **finiteness** is not **absolute**.

What to Do?

Potential Fixes

(Non-)
Absoluteness
of ℕ

Hongyu
Zhu

Survey

Defining ℕ

Is There a
Problem?

What to
Do?

Bibliography

Finiteness:

- Strengthen/Change PA/ZFC

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- Restrict possible universes/Fix a possible universe
- Focus only on common truths

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If we decide on a complete theory/single universe, what would be a nice (workable) description? (cf. Incompleteness Theorems)

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If we decide to live in an incomplete and non-absolute world, what to do about the undecided?

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- "Incremental Expansion"? (e.g. Large Cardinals)

If we decide to live in an incomplete and non-absolute world, what to do about
the undecided?

- "Classification Theory"? (e.g. CH)

Upside Down?

(Non-)
Absoluteness
of ℕ

Hongyu
Zhu

Survey

Defining ℕ

Is There a
Problem?

What to
Do?

Bibliography

If finiteness is at stake, can we employ any mathematical argument at all (let alone resolving the issue)?

Bibliography

(Non-)
Absoluteness
of ℕ

Hongyu
Zhu

Survey

Defining ℕ

Is There a
Problem?

What to
Do?

Bibliography

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Thank you for listening!

Comments, supplements, and criticisms are welcome.