On the Bounded Axiomatizability of Theories and Types

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Table of Contents

Arithmetic is “bad.” For example:

- True arithmetic is model-theoretically as wild as possible;
- Peano arithmetic has no non-standard computable models;
- Peano arithmetic has no recursively axiomatizable completion.

Another perspective: Consider the subsystems $I\Sigma_n$ of PA, where we restrict induction to Σ_n^0 formulas. It is known that PA is stronger than every $I\Sigma_n$.
Generalizing this phenomenon, we would like to understand when a theory is (not) captured by its restrictions “of lower complexity.”
When “of lower complexity” means a single formula, we get the notion of finite axiomatizability, which is well-studied. In the PA example, Σ_n^0 is the notion of complexity, and we would like to generalize to arbitrary theories.

Introduction: Preliminaries

Definition

- $\forall_0 = \exists_0$ is the set of quantifier-free formulas.
- The set of $\forall_{n+1}$-*formulas* consists of all formulas of the form $\forall x_1 \ldots \forall x_n\ \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ where $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is an $\exists_n$-formula.
- The set of $\exists_{n+1}$-*formulas* consists of all formulas of the form $\exists x_1 \ldots \exists x_n\ \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ where $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a $\forall_n$-formula.
- If a theory has an axiomatization consisting entirely of $\forall_n$-formulas, then we say it is $\forall_n$-*axiomatizable* (similarly for $\exists_n$).

Remark

Over PA, $\forall_n = \Pi^0_n$ and $\exists_n = \Sigma^0_n$ for $n \geq 1$ (due to the MRDP theorem).

Bounded Axiomatizability: Definition and Examples for Theories

Definition

A theory is *boundedly axiomatizable* (or *bounded* for short) if it is $\forall_n$-axiomatizable for some finite n. Otherwise, it is not boundedly axiomatizable (*unbounded*).

Example (bounded theories)

- $\mathrm{Th}(\mathbb{Q}, <)$, theory of groups/rings/fields, etc.
- More generally, any finitely axiomatizable theory is bounded.
- Every model complete theory is $\forall_2$-axiomatizable, thus bounded.
- A bounded theory that is not finitely axiomatizable: Consider $\mathcal{L} = \{P_n \mid n \in \omega\}$ with only unary predicates, and let T_ω be the (complete) theory saying the P_n's are all infinite and pairwise disjoint.
- Apply Marker extension to T_ω, making P_3 $\forall_3$, so that we get a bounded but not model complete theory.

Theories we expect to be complicated are indeed unbounded.

Observation (Rabin)

All consistent extensions of PA, in particular PA and TA, are unbounded.

For complete theories, it is easy to become unbounded. Say a theory is *sequential* if it is able to "encode finite sequences." (It directly interprets adjunctive set theory.)

Theorem (Enayat, Visser)

Every complete sequential theory in a finite language is unbounded. This includes, for example, completions of PA^- *and* ZFC*.*

Bounded Axiomatizability: Unboundedness of Theories

Unboundedness also arises naturally from descriptive set theory.
By identifying each countable structure (having domain ω) with its atomic diagram, we can view it as an element of 2^ω. Then we can talk about the Borel complexity of $\mathrm{Mod}(T)$, the set of (countable) models of T.
It's not hard to see that for a first-order theory T, $\mathrm{Mod}(T)$ is always (at most) $\mathbf{\Pi}^0_\omega$.

Theorem (Andrews, Gonzalez, Lempp, Rossegger, Z.)

A complete theory T is unbounded if and only if $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_\omega$-complete.

Corollary (Andrews, Gonzalez, Lempp, Rossegger, Z.)

The set of models of TA*, or any completion of* PA*, is $\mathbf{\Pi}^0_\omega$-complete. (In fact, so is* $\mathrm{Mod}(\mathrm{PA})$*.)*

Bounded Axiomatizability: Definition and Examples for Types

Definition

A type $p(\overline{x})$ is *bounded* (or *unbounded*) if it is so when viewed as a theory (after extending the language with constants $\overline{a}$ matching $\overline{x}$).

Example (bounded types)

- All isolated types (over a bounded theory) are bounded.
- If T is model complete (in particular if it has QE), then every type over T is bounded.

Example (unbounded types)

- Consider the Marker extension of T_ω (infinitely many infinite sorts) where each P_n becomes $\forall_n$. Then the non-isolated 1-type ($\neg P_n(x)$ for all n) is unbounded.
- One can code unbounded types in arithmetic, or set theory.

The Theory-to-Type Conjecture: The Conjecture

It seems that unbounded types show up mostly in unbounded theories. This observation leads to the following conjecture:

Conjecture (the theory-to-type conjecture)

If a complete theory T is bounded, then every (complete) type of T is also bounded.

In the context of infinitary logic, the conjecture is better understood and motivated. This is not entirely surprising: Types are more structural (as is infinitary logic), and they are naturally infinitary objects.

The Theory-to-Type Conjecture: Infinitary Logic

We extend first-order logic to the infinitary logic $\mathcal{L}_{\omega_1\omega}$ by adding countable conjunctions and disjunctions, while quantification remains limited to finitely many variables. Countable conjunctions and disjunctions will be denoted by special symbols: $\bigwedge, \bigvee$.

The following hierarchy is analogous to the finitary hierarchy, and we treat countable conjunctions/disjunctions as universal/existential quantifiers.

Definition

- $\Sigma_0^{\text{in}} = \Pi_0^{\text{in}}$ consists of all *finitary* quantifier-free formulas.
- For a countable $\alpha > 0$ and formulas φ, ψ, say φ is $\Sigma_\alpha^{\text{in}}$ and ψ is Π_α^{in} if

$$\varphi \equiv \bigvee_{i\in I} \exists \overline{x} \chi_i(\overline{x}), \psi \equiv \bigwedge_{i\in I} \forall \overline{x} \theta_i(\overline{x}),$$

  where each χ_i is Π_β^{in} and each θ_i is Σ_β^{in} for some $\beta < \alpha$.

The Theory-to-Type Conjecture: Infinitary Logic

Theorem (Scott Isomorphism Theorem)

Every countable structure $\mathcal{M}$ has an $\mathcal{L}_{\omega_1\omega}$ Scott sentence, i.e. a sentence φ which characterizes $\mathcal{M}$ up to isomorphism among countable structures.

An infinitary analogue of the theory-to-type conjecture has been established:

Theorem (Montalbán)

Given a countable structure $\mathcal{A}$ and a countable α, the following are equivalent:

- *$\mathcal{A}$ has a $\Pi^{\mathrm{in}}_{\alpha+1}$ Scott sentence.*
- *Every automorphism orbit of $\mathcal{A}$ is $\Sigma^{\mathrm{in}}_{\alpha}$-definable.*

The least such α is the Scott rank of α.

Informally, taking $\alpha = n < \omega$, this means "the complete infinitary theory of $\mathcal{A}$ is $\forall_{n+1}$" exactly when "all infinitary types of $\mathcal{A}$ are $\exists_n$."

The Theory-to-Type Conjecture: About the ω-Vaught's Conjecture

The structural implications of the theory-to-type conjecture is more evident in relation to the ω-Vaught's conjecture, an effectivization of Vaught's conjecture proposed by Gonzalez and Montalbán.

Definition

The *Vaught ordinal* of φ is the least β such that: either φ has only countably many models, all with Scott rank below β; or there are uncountably many models of φ pairwise distinguished by a Π^{in}_{β} formula.

Conjecture (ω-Vaught's conjecture)

Every $\varphi \in \Pi^{\text{in}}_{\alpha}$ has Vaught ordinal at most $\alpha + \omega$.

Now suppose φ is (the countable conjunction of) a first-order bounded (small) theory T, and that (for simplicity) φ has countably many models.

A Scott sentence for $\mathcal{M} \vDash \varphi$ is often given in terms of types. For example, if $\mathcal{M}$ is the prime model, then the Scott sentence simply says all non-isolated types are omitted (in addition to φ). However, unbounded types are Π^{in}_{ω} in general, and when T is bounded this is not good enough for the ω-Vaught's conjecture. This is why we need the theory-to-type conjecture.

More concretely, when T is ω-stable, the descriptions required by the ω-Vaught's conjecture is almost there (cf. Shelah-Harrington-Makkai), modulo the complexity of types.

The Theory-to-Type Conjecture: About the ω-Vaught's Conjecture

	Theory T	Types p	Description	Required	Good?
General	α	β	$< \beta + \omega$	$< \alpha + \omega$	If $\beta + \omega \leq \alpha + \omega$
T unbounded	ω	$\leq \omega$	$< \omega + \omega$	$\omega + \omega$	Yes
T bounded, p bounded	$< \omega$	$< \omega$	$< \omega$	$< \omega$	Yes
T bounded, p unbounded	$< \omega$	ω	$< \omega + \omega$	$< \omega$	No!

The theory-to-type conjecture rules out the last case.

Theorem (Z., in progress)

The theory-to-type conjecture for ω-stable theories implies the ω-Vaught's conjecture for ω-stable theories.

The Theory-to-Type Conjecture: Resolving the Conjecture

Theorem (Z.)

The theory-to-type conjecture fails in general: There is a complete, strictly superstable theory T which is bounded (in fact $\forall_1$-axiomatizable) but has unbounded types.

The idea is to use trees, where complexity can be coded into types; Then we hide such complexity, so that the theory cannot find out without parameters.

Question

Is the theory-to-type conjecture true for ω-stable theories?

Thank you for listening!

References I

[1] Uri Andrews, David Gonzalez, Steffen Lempp, Dino Rossegger, and Hongyu Zhu. “The Borel complexity of the class of models of first-order theories”. In: *Proc. Amer. Math. Soc.* 153.9 (2025), pp. 4013–4024. ISSN: 0002-9939,1088-6826. DOI: 10.1090/proc/17308. URL: https://doi.org/10.1090/proc/17308.

[2] Ali Enayat and Albert Visser. “Incompleteness of boundedly axiomatizable theories”. In: *Proceedings of the American Mathematical Society* 152.11 (Sept. 20, 2024), pp. 4923–4932. ISSN: 0002-9939, 1088-6826. DOI: 10.1090/proc/16975. URL: https://www.ams.org/proc/2024-152-11/S0002-9939-2024-16975-6/.

References II

[3] David Gonzalez and Antonio Montalbán. "The ω-Vaught's conjecture". In: *Transactions of the American Mathematical Society* 376.8 (May 31, 2023), pp. 5989–6008. ISSN: 0002-9947, 1088-6850. DOI: 10.1090/tran/8950. URL: https://www.ams.org/tran/2023-376-08/S0002-9947-2023-08950-4/.

[4] Antonio Montalbán. "A robuster Scott rank". In: *Proceedings of the American Mathematical Society* 143.12 (Apr. 14, 2015), pp. 5427–5436. ISSN: 0002-9939, 1088-6826. DOI: 10.1090/proc/12669. URL: http://www.ams.org/proc/2015-143-12/S0002-9939-2015-12669-X/.

[5] Saharon Shelah, Leo Harrington, and Michael Makkai. "A proof of Vaught's conjecture for ω-stable theories". In: *Israel Journal of Mathematics* 49 (1984), pp. 259–280. URL: https://doi.org/10.1007/BF02760651.

[6] Hongyu Zhu. "A Complete Bounded Theory with Unbounded Types". In: *The Journal of Symbolic Logic* (2026). Published online ahead of print. DOI: 10.1017/jsl.2026.10208. URL: https://doi.org/10.1017/jsl.2026.10208.

[7] Hongyu Zhu. *The Type ω-Vaught's Conjecture.* In preparation.