A Complete Bounded Theory with Unbounded Types

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Background: Formulas

- $\forall_0 = \exists_0$ is the set of quantifier-free formulas.
- The set of $\forall_{n+1}$-*formulas* consists of all formulas of the form $\forall x_1 \ldots \forall x_n \; \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ where $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is an $\exists_n$-formula.
- The set of $\exists_{n+1}$-*formulas* consists of all formulas of the form $\exists x_1 \ldots \exists x_n \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ where $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a $\forall_n$-formula.
- If a theory has an axiomatization consisting entirely of $\forall_n$-formulas, then we say it is $\forall_n$-*axiomatizable* (similarly for $\exists_n$).

Background: Descriptive Complexity

Throughout, we work in a countable language $\mathcal{L}$. All countable structures are assumed to have domain ω.

Notation

Fix a listing $\{\varphi_i \mid i \in \omega\}$ of all atomic $(\mathcal{L} \cup \omega)$-sentences, where elements of ω are viewed as constants. We identify any countable $\mathcal{L}$-structure $\mathcal{A}$ with its atomic diagram $\mathcal{D}(\mathcal{A}) \in 2^\omega$, i.e. $\mathcal{D}(\mathcal{A})(i) = 1$ if $\mathcal{A} \vDash \varphi_i$ and 0 otherwise.
$\mathrm{Mod}(T) \subseteq 2^\omega$ is the set of all countable models of T.

Definition

By the *descriptive complexity* of a theory T we mean the descriptive complexity of $\mathrm{Mod}(T) \subseteq 2^\omega$. For example, we say T is $\mathbf{\Pi}^0_n$ if $\mathrm{Mod}(T)$ is a $\mathbf{\Pi}^0_n$ subset of 2^ω. The same applies to a type (or a partial type) $p(\overline{x})$ in language $\mathcal{L}$, by viewing $p(\overline{x})$ as the theory $p(\overline{a})$ in language $\mathcal{L} \cup \{\overline{a}\}$ where $\overline{a}$ is a new set of constants (matching the length of $\overline{x}$).

Background: Boundedness

Definition

A theory is *boundedly axiomatizable* (or *bounded* for short) if it is $\forall_n$-axiomatizable for some finite n. Otherwise, it is not boundedly axiomatizable (or *unbounded* for short).

Fact (Enayat, Visser)

Any consistent bounded sequential theory in a finite language must be incomplete.

Fact (Andrews, Gonzalez, Lempp, Rossegger, Z.)

A complete theory T is unbounded if and only if $\mathrm{Mod}(T)$ *is* $\mathbf{\Pi}^0_\omega$*-complete (which is the maximum possible by López-Escobar).*

Background: Main Question

Question

If a theory T is bounded, is every (complete) type of T also bounded?

An analogous theorem in infinitary logic ($\mathcal{L}_{\omega_1\omega}$, where countable conjunctions and disjunctions are allowed):

Theorem (Montalbán)

Given a countable structure $\mathcal{A}$ and a countable α, the following are equivalent:

- *$\mathcal{A}$ has a $\Pi^{\text{in}}_{\alpha+1}$ Scott sentence.*
- *Every automorphism orbit of $\mathcal{A}$ is $\Sigma^{\text{in}}_{\alpha}$-definable.*

Informally, taking $\alpha = n < \omega$, this means "the complete infinitary theory of $\mathcal{A}$ is $\forall_{n+1}$" exactly when "all infinitary types of $\mathcal{A}$ are $\exists_n$." Thus, the boundedness of the the theory and the types align in the infinitary case.

Background: Main Question

The main question also relates to the effective analysis of Vaught's Conjecture.

For example, if T has countably many countable models and is model-theoretically tame (e.g. ω-stable), each model can be characterized using types. (A simple example is the prime model, where only the isolated types are realized.) This allows us to write down a *Scott sentence*, i.e. an $\mathcal{L}_{\omega_1\omega}$-sentence characterizing the model up to isomorphism. However, the precise complexity of the Scott sentence (e.g. the least α such that the sentence is $\Pi^{\mathrm{in}}_{\alpha}$) in general depends on the boundedness of types.

Such distinctions show up when working with the ω-Vaught's Conjecture of Gonzalez and Montalbán, where effective analysis is also conducted when T has uncountably many models.

Background: Main Result

In addition, usually when unbounded types show up, the underlying theory is already unbounded. (For example, in arithmetic or in Marker extensions.)
On the other hand, our main result states that:

Theorem (Z.)

There is a complete theory T which is bounded (in fact $\forall_1$-axiomatizable) but has unbounded types. In addition, it is strictly superstable.

The idea is to use trees, where complexity can be coded into types; Then we hide such complexity, so that the theory cannot find out without parameters.

Constructing the Theory: Overview

We will follow the outline below:

- Introduce a base theory T_0 of trees;
- Build the actual theory T;
- Prove the completeness of T using EF-games;
- Analyze the types for stability and unboundedness properties.

The most difficult part is completeness. For example, T cannot admit quantifier elimiation/be model complete/etc., since the existence of an unbounded type requires, for each n, a formula that is not equivalent to any $\forall_n$-formula.

Constructing the Theory: Base Language

Definition

Let $\mathcal{L}_0 = \{P_i \mid i \in \omega\} \cup \{<_i \mid i \in \omega\}$, where P_i are unary predicates and $<_i$ are binary relations.

Notation

Let P denote $\bigvee_{i\in\omega} P_i$, and $<$ denote $\bigvee_{i\in\omega} <_i$. (Not in general definable.)

The language is like "leveled" trees: We want to say a node x is on the tree with $P(x)$, and say x is the predecessor of y with $x < y$. In $\mathcal{L}_0$, we need to know the "level" n of x, and then say $P_n(x)$ or $x <_n y$, respectively.
This stratification will be useful to making our trees "pseudofinite."

Convention

Trees grow up; Roots are on level 0.

For example, if x is a root and y is a successor of x, then $P_0(x), P_1(y)$, and $x <_0 y$.

Constructing the Theory: Base Theory

The theory T_0 basically says the model is a forest (disjoint union of trees); and possibly points outside of P (by compactness).

Definition

Let T_0 be the $\mathcal{L}_0$-theory that says (for every $i, j \in \omega$ with $i \neq j$):

- $P_i \cap P_j = \varnothing$;
- $<_i \subseteq P_i \times P_{i+1}$;
- (Existence of predecessors) $\forall x \in P_{i+1}\, \exists y \in P_i\, (y <_i x)$;
- (Uniqueness of predecessors) $\forall x \forall x' \forall y\, ((x <_i y \wedge x' <_i y) \to x = x')$.

Notice that T_0 is $\forall_2$, but we can make it $\forall_1$ by replacing $<_i$ with a function symbol. The choice here is to make the language relational.

Constructing the Theory: Basic Properties of T_0

Observation

T_0 has $\aleph_0$ finite models (up to isomorphism). In fact, it has finitely many models of each finite cardinality n.

Observation

Any (not necessarily finite) disjoint union of models of T_0 remains a model of T_0.

Constructing the Theory: the Actual Theory

We want to allow arbitrary trees in our models while having a complete theory. The workaround is to throw in all finite forests, so the theory has no access to specific behaviors on the tree without seeing the root.
List the finite models of T as $\{\mathcal{M}_i\}_{i<\omega}$. For each $j < \omega$, let C_i^j be a set of new constants of size $|M_i| < \omega$. Let $C = \bigcup_{i,j\in\omega} C_i^j$.

Definition

Let $\widetilde{\mathcal{L}} = \mathcal{L}_0 \cup C$. Let T be the $\widetilde{\mathcal{L}}$-theory that says:

- T_0;
- The constants in C are pairwise distinct;
- The $\mathcal{L}_0$-substructure with domain C_i^j is isomorphic to $\mathcal{M}_i$;
- For each $k \in \omega$: For all x, y, if $x \in C_i^j$ and $x <_k y \vee y <_k x$, then $y \in C_i^j$.

Again, this is $\forall_2$, and can be made $\forall_1$ using function symbols.

Constructing the Theory: Basic Properties of T

Observation

There is a unique model $\mathcal{C} \models T$ whose domain is equal to $C^{\mathcal{C}}$, i.e. every element is (the interpretation of) a constant. In addition, $\mathcal{C}$ embeds into every model of T.

Observation

As $\mathcal{L}_0$-structures, $\mathcal{C} \cong \bigsqcup_i \mathcal{M}_i^\omega$ (where $\mathcal{A}^\omega$ is the disjoint union of ω copies of $\mathcal{A}$).

Observation

For all $\mathcal{M} \models T$, $\mathcal{M} \cong \mathcal{C} \sqcup (\mathcal{M} \backslash \mathcal{C})$ over $\mathcal{L}_0$ (where $\mathcal{M} \backslash \mathcal{C}$ is the $\mathcal{L}_0$-substructure of $\mathcal{M}$ with domain $M \backslash C$).

The idea of the completeness is that whatever trees living in $\mathcal{M} \backslash \mathcal{C}$ will be $\mathcal{L}_0$-pseudofinite, and then $\mathcal{C} \cong \bigsqcup_i \mathcal{M}_i^\omega$ absorbs all finite trees, leading to $\mathcal{M} \equiv \mathcal{C}$.

Completeness: EF-games

We will use the EF-games as follows.

Definition

Fix a language $\mathcal{L}$ and $\mathcal{L}$-structures $\mathcal{M}_0, \mathcal{M}_1$. The game $EF_k^{\mathcal{L}}(\mathcal{M}_0, \mathcal{M}_1)$ is the following length-k game between players $\forall$ and $\exists$: At each step,

- $\forall$ chooses an element from M_0 or M_1;
- $\exists$ then chooses an element from the other structure.

In the end, collect all chosen elements $\overline{m}_i$ from M_i, and $\exists$ wins the play if and only if there is an isomorphism $f : \langle \overline{m}_0 \rangle_{\mathcal{M}_0} \to \langle \overline{m}_1 \rangle_{\mathcal{M}_1}$ identifying $\forall$'s choice at each step with $\exists$'s corresponding choice.

Completeness: EF-games

Definition

For all $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$:

- $\mathcal{M} \equiv^{EF}_{\mathcal{L},k} \mathcal{N}$ if $\exists$ has a winning strategy in $EF^{\mathcal{L}}_k(\mathcal{M}, \mathcal{N})$.
- $\mathcal{M} \equiv^{EF}_{\mathcal{L}} \mathcal{N}$ if $\mathcal{M} \equiv^{EF}_{\mathcal{L},k} \mathcal{N}$ for every finite k, i.e. $\exists$ wins every $EF^{\mathcal{L}}_k(\mathcal{M}, \mathcal{N})$ of finite length.

The following theorem can be found in standard texts on model theory.

Theorem

If $\mathcal{M} \equiv^{EF}_{\mathcal{L}} \mathcal{N}$ in a finite language $\mathcal{L}$, then $\mathcal{M} \equiv \mathcal{N}$ in the same language.
As a result, if $\mathcal{M} \equiv^{EF}_{\mathcal{L}'} \mathcal{N}$ in every finite $\mathcal{L}' \subseteq \mathcal{L}$, then $\mathcal{M} \equiv \mathcal{N}$ over $\mathcal{L}$.

Completeness: Pseudofiniteness

Idea of the Approximation: Given $\mathcal{M}, \mathcal{N} \models T$, we want to show $\mathcal{M} \equiv^{EF}_{\mathcal{L}',k} \mathcal{N}$ where $\mathcal{L}' \subseteq \widetilde{\mathcal{L}}$ is a finite sublanguage and $k < \omega$. We may assume $\mathcal{L}'$ "sees up to" level h for some finite h, i.e. it does not contain $P_l, <_{l-1}$ for $l > h$.
Under such assumptions, all trees seem to have height at most h. In addition, since we know the game has length k, having ω leaves that "look the same" is no different than having k of them. So we arrive at a $\leq k$-branching tree with height $\leq h$, which is clearly finite.
More formally:

Definition

An *h-forest* (or h-tree if there's only one root) is a model of T_0 restricted to level h (i.e. to $\mathcal{L}_h = \{P_l, <_{l-1} \mid l \leq h\}$).

Completeness: Pseudofiniteness

Definition

The *k-bounded coloring* of an h-forest $\mathcal{M}$ is the function Λ defined on M as follows: given $x \in M$, $\Lambda(x) = \langle \Lambda_0(x), \Lambda_1(x) \rangle$ where

- $\Lambda_0(x)$ is the level of x (or -1 if not in $P_{\leq h}$).
- $\Lambda_1(x)$ is a set of pairs $\langle \lambda_i, n_i \rangle$. It is defined inductively, from the leaves down to the root, as follows:
  - Let $\Lambda_1(x)$ be the set of all pairs $\langle \lambda, n \rangle$, where λ is the *color* of a successor y of x (i.e. $\lambda = \Lambda(y)$), and $n = \min(k, m)$ where m is the number of successors of x with color λ.
  - In particular, $x \in P_{\leq h}$ is a leaf if and only if $\Lambda_1(x) = \varnothing$.

Remark

Recall that h-forests have height $\leq h$, so every node will indeed have a color.

Completeness: Pseudofiniteness

Proposition

For every possible (k, h)-color λ of a root node, there is a finite h-tree $\mathcal{Y}_\lambda$ whose root has color λ.

Proof.

By definition. □

Proposition

Fix $0 < h < \omega, n \in \omega$. For every h-tree $\mathcal{M}$, we have $\mathcal{M} \equiv^{EF}_{\mathcal{L}_{h,n}} \mathcal{Y}_\lambda$ where λ is the $(n(h+1), h)$-color of the root of $\mathcal{M}$.

Proof.

Use an inductive color-matching strategy, noting that we may need up to $h + 1$ steps to verify that a new node played by $\forall$ is on the same tree as a previously played node. □

Completeness: Pseudofiniteness

Definition

$\mathcal{M}$ is *pseudofinite* if every formula satisfied by $\mathcal{M}$ is satisfied by a finite structure.

Remark

Other equivalent definitions:

- $\mathcal{M}$ satisfies the common theory of all finite structures;
- $\mathcal{M}$ is elementarily equivalent to an ultraproduct of finite structures.

The following theorem is well-known.

Theorem

For every $\mathcal{L}$-formula $\varphi(\overline{x})$, there exists $n < \omega$ such that for all $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$, if $\mathcal{M} \equiv^{EF}_{\mathcal{L},n} \mathcal{N}$, then $\mathcal{M} \models \varphi \iff \mathcal{N} \models \varphi$.

One can basically take n to be the quantifier rank of the formula.

Completeness: Pseudofiniteness

Theorem

Finite-height trees (e.g. in the language of directed graphs) are pseudofinite.

Proof.

If a tree has finite height, it is bi-interpretable with an h-forest carrying the same information. By the last proposition, h-forests are pseudofinite. □

Theorem

T is complete.

Proof.

Fix $\mathcal{M} \models T$ and $0 < h < \omega$. For all k, there exists $\mathcal{F}_k$ such that $\mathcal{M} \backslash \mathcal{C} \equiv^{EF}_{\mathcal{L}_h, k} \mathcal{F}_k$ with $\mathcal{F}_k$ a disjoint union of finite trees. Thus $\mathcal{C} \cong \mathcal{F}_k \sqcup \mathcal{C}$, so (with a bit of extra work) $\mathcal{M} \equiv^{EF}_{\mathcal{L}_h, k} \mathcal{F}_k \sqcup \mathcal{C} \cong \mathcal{C}$. This implies $\mathcal{M} \equiv \mathcal{C}$, so $T = \mathrm{Th}(\mathcal{C})$. □

Analysis of Types: Counting

Remark

There is a first-order formula saying that the (k,h)-color of x is a certain color λ.

Using a color-matching argument with parameters, we can show that:

Proposition

Any n-type $p(\overline{x})$ over T is completely determined by: (1) Whether or not each x_i is a constant; (2) For each i, the unique n such that $x_i \in P_n$ (or the nonexistence thereof); (3) For each k,h, the (k,h)-color of each x_i and ancestors; (4) For any $x_i, x_j \in \overline{x}$, the unique (u,v) such that the u-th predecessor of x_i is equal to the v-th predecessor of x_j (if it exists).

This characterization extends to types over a set S.

Corollary

T is strictly superstable.

Analysis of Types: Unbounded Types

To build an unbounded type, we follow a 2-step process: (1) Build $\mathbf{\Pi}^0_n$-hard types; (2) Combine them into a single $\mathbf{\Pi}^0_\omega$-complete type.
For the $\mathbf{\Pi}^0_n$-hard types, we use the *back-and-forth trees* $\mathcal{A}_k, \mathcal{E}_k$ defined by Hirschfeldt and White.
Base case: $\mathcal{A}_1$ = a single root, $\mathcal{E}_1$ = a root with ω leaf successors.
Inductive step: "…" below mean ω copies.

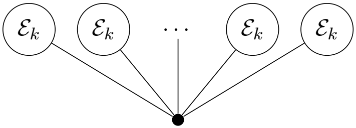

Figure: $\mathcal{A}_{k+1}$

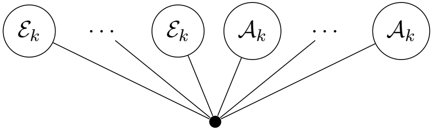

Figure: $\mathcal{E}_{k+1}$

Key properties: $\mathcal{A}_k$ is $\mathbf{\Pi}^0_k$-hard, $\mathcal{E}_k$ is $\mathbf{\Sigma}^0_k$-hard, and a $\forall_k$-formula distinguishes them. (Hirschfeldt, White; Csima, Deveau, Harrison-Trainor, Mahmoud)

Analysis of Types: Unbounded Types

To build a $\mathbf{\Pi}^0_\omega$-complete (thus unbounded) type, we combine the $\mathcal{A}_k$'s into a single tree where $\mathcal{A}_k$'s position is definable relative to the root; For example as follows.

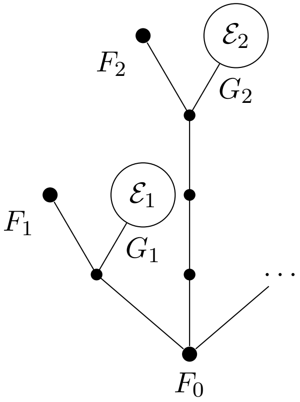

Figure: $\mathcal{N}$

Further Questions

- Is there a complete bounded ω-stable theory with unbounded types?
    - This relates to the ω-Vaught's Conjecture for ω-stable theories.
- Is there a complete bounded theory with unbounded types having only countably many countable models?
- Is there a $\exists_2$-axiomatizable complete *relational* theory with unbounded types?
    - Our example is either $\forall_2$ relational, or $\forall_1$ non-relational.
    - Relational $\forall_1$ or $\exists_1$ theories are impossible.
- Is there a similar construction for other structures, like graphs?

Thank you for listening!

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