

Comparing Lightface and Boldface Definitions

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Introduction: The setting

Definition

A *Polish space* is a separable, completely metrizable topological space.

Example

- Let $2^{\mathbb{N}}$ be the space of all infinite binary sequences. The open sets are generated by the basic clopen sets of the following form: ($\sigma \in 2^{<\mathbb{N}}$ is a finite binary string)

$$[\sigma] := \left\{ X \in 2^{\mathbb{N}} \mid \sigma \prec X \right\}.$$

It is a Polish space: Separability is clear, and a complete metric is given by: $d(X, Y) = 2^{-\min\{i \mid X(i) \neq Y(i)\}}$ if $X \neq Y$, and 0 if $X = Y$.

- $\mathbb{N}^{\mathbb{N}}$, the space of all infinite sequences of naturals, is a Polish space by the same construction.

Introduction: The boldface hierarchy

We work in a fixed Polish space ($2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$), and consider the topological complexity of its Borel subsets.

Definition

- Σ_1^0 = open sets, Π_1^0 = closed sets, Δ_1^0 = clopen sets.
- Σ_α^0 = countable union of $\Pi_{<\alpha}^0$ sets.
- Π_α^0 = countable intersection of $\Sigma_{<\alpha}^0$ sets.
- A subset is Δ_α^0 iff it is both Σ_α^0 and Π_α^0 .

Remark

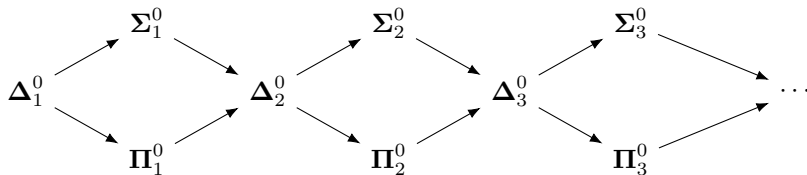
- $\bigcup_{\alpha < \omega_1} \Pi_\alpha^0 = \text{Borel}$.
- $\Sigma_2^0 = \bigcup(\text{closed}) = F_\sigma$. Similarly, $\Pi_2^0 = \bigcap(\text{open}) = G_\delta$.

Introduction: The boldface hierarchy

Fact

$$\Sigma_{\alpha}^0, \Pi_{\alpha}^0 \subseteq \Delta_{\alpha+1}^0.$$

Below is a diagram illustrating the Borel hierarchy.



Introduction: The lightface hierarchy

While an open set is conceptually simple, it might not have easy concrete descriptions. One way to describe it explicitly is to list all the strings that generate a basic open subset of it, and for nice open sets this can be done *effectively*:

Definition

A set of natural numbers/finite strings/codes/... is *computably enumerable (c.e.)* if it can be listed by a computer program.

Definition

An open set U is *effectively open* if it is of the form

$$\bigcup_{\sigma \in W} [\sigma]$$

where W is a c.e. set of strings. A *code* for U is a code for a program listing W .

Similar definitions can be made for *effectively closed sets*.

Introduction: The lightface hierarchy

Now we iterate the process like the boldface hierarchy, additionally keeping track of the codes.

Definition

- Σ_1^0 = effectively open sets, Π_1^0 = effectively closed sets.
- Σ_α^0 = effective union of $\Pi_{<\alpha}^0$ sets. A code gives a program that lists the codes of constituent sets uniformly.
- Π_α^0 = effective intersection of $\Sigma_{<\alpha}^0$ sets. A code gives a program that lists the codes of constituent sets uniformly.
- A subset is Δ_α^0 iff it is both Σ_α^0 and Π_α^0 . A code consists of a Σ_α^0 -code and a Π_α^0 -code.

While the boldface hierarchy stops at ω_1 , the lightface version stops at ω_1^{CK} , the least non-computable ordinal (which is countable).

Introduction: The lightface hierarchy

Example

- Consider the set consisting of only the constant zero sequence. Its complement is effectively open, and it is effectively closed.
- In general, any eventually constant sequence is a Π_1^0 singleton.
- The collection of eventually constant sequences U_{ec} is Σ_2^0 .

Notice that U_{ec} can be defined by a formula with 2 “alternations” of (number) quantifiers, starting with $\exists$:

$$X \in U_{ec} \iff \exists n \exists i \forall m > n \ X(m) = i.$$

This translates to an expression of U_{ec} as a Σ_2^0 set:

$$U_{ec} = \bigcup_{n,i} \bigcap_{m>n} C_{n,i,m}, \quad C_{n,i,m} = \{X \mid X(m) = i\}.$$

Introduction: The lightface hierarchy

The correspondence between number quantifiers and countable union/intersections leads to the following theorem/observation.

Theorem

The following are equivalent for all $n \in \mathbb{N}$ and subsets U .

- *U can be defined by a formula with n alternations of number quantifiers, starting with $\exists$.*
- *U is Σ_n^0 .*

Similarly for Π_n^0 (where formulas start with $\forall$).

Introduction: Beyond the Borel sets

It is crucial in the previous theorem that the quantifiers are not set quantifiers, since these can easily make sets non-Borel.

Definition

A set $U \subseteq \mathbb{N}^{\mathbb{N}}$ is *analytic*, or Σ_1^1 , if it is the projection of a Borel (equivalently closed) set. It is *coanalytic*, or Π_1^1 , if its complement is analytic.

Definition

A set $U \subseteq \mathbb{N}^{\mathbb{N}}$ is Σ_1^1 if it has a definition of the form $X \in U \iff \exists Y \varphi(X, Y)$, where φ is a Π_1^0 definition. It is Π_1^1 if its complement is Σ_1^1 .

Example

WF, the collection of all well-founded trees (coded by infinite sequences), is Π_1^1 and Π_1^1 .

Comparing the hierarchies: Cardinalities

One clear difference between the two hierarchies is their cardinalities.

Observation

For all $\alpha \geq 1$, there are $2^{\aleph_0}$ many Σ_α^0 sets but only $\aleph_0$ many Σ_α^0 sets.

Proof.

Note that there are $2^{\aleph_0}$ many open sets and thus $2^{\aleph_0}$ many Borel sets. On the other hand, each lightface definition corresponds to a program (or a formula), and there are only $\aleph_0$ programs (or formulas). □

Comparing the hierarchies: Relativization

It is nevertheless possible to obtain lightface descriptions of sets in the boldface hierarchy, relative to an additional *oracle*: Suppose U is Σ_α^0 . Let $X \in 2^\mathbb{N}$ (or $\mathbb{N}^\mathbb{N}$) contain all the information used to construct U (via union/intersection), then there is a “ Σ_α^0 -code” for U using X as a parameter.

Definition

Let α be an X -computable ordinal. U is Σ_α^0 *relative to* X , or $\Sigma_\alpha^0(X)$, if it can be described by a Σ_α^0 -code that can access X as an oracle.

When $\alpha = n \in \mathbb{N}$, U can (equivalently) be defined by a Σ_n^0 formula with X as an additional parameter.

Comparing the hierarchies: Relativization

Here is a formal way of saying “boldface = lightface relative to an oracle.”

Theorem

For any computable ordinal α ,

$$\Sigma_{\alpha}^0 = \bigcup_{X \in 2^{\omega}} \Sigma_{\alpha}^0(X), \Pi_{\alpha}^0 = \bigcup_{X \in 2^{\omega}} \Pi_{\alpha}^0(X), \Delta_{\alpha}^0 = \bigcup_{X \in 2^{\omega}} \Delta_{\alpha}^0(X).$$

So when a set U is Σ_{α}^0 but not Σ_{α}^0 , the collection of oracles X such that U is $\Sigma_{\alpha}^0(X)$ gives a concrete measure of the difference between lightface and boldface complexities.

Comparing the hierarchies: Subsets with high lightface complexity

Now we look at examples where the boldface and (unrelativized) lightface complexities do not agree.

Example

Singletons $\{X\}$ are clearly Π_1^0 , but not all of them are Π_1^0 .

- If X is computable (for example, any constant sequence), then $\{X\}$ is Π_1^0 .
- If $X = 0'$ is the halting problem (which is not computable), then $\{X\}$ is Π_2^0 .
More generally, if $X \leq_T 0^{(\alpha)}$, then $\{X\}$ is $\Pi_{1+\alpha}^0$.
- If $X = 0^{(\omega)} = \bigoplus_{n \in \mathbb{N}} 0^{(n)}$, then $\{X\}$ is still Π_2^0 ! More generally, $\{0^{(n)}\}$ is in fact always Π_2^0 .
- If $X \leq_T 0^{(n)}$ is weakly n -random or weakly n -generic, then $\{X\}$ is strictly Π_{1+n}^0 .

In all cases, relative to the oracle X the complexity drops down to Π_1^0 .

Comparing the hierarchies: Subsets with high lightface complexity

The next collection of examples are countable sets $U = \{X_i \mid i \in \mathbb{N}\}$, which are Σ_2^0 . It becomes Σ_2^0 relative to any oracle X that uniformly computes $\{X_i\}_{i \in \omega}$.

Example

- If $\{X_i\}_{i \in \omega}$ is uniformly computable (for example, all constant sequences), then $\{X\}$ is Σ_2^0 .
- If $\{X_i\}_{i \in \omega}$ are exactly the computable sets, then U is Σ_3^0 .
- If $\{X_i\}_{i \in \omega}$ are exactly the arithmetical sets (X such that $X \leq_T 0^{(n)}$ for some $n \in \mathbb{N}$), then U is strictly Σ_ω^0 . $0^{(\omega)}$ suffices to compute those sets.
- If $\{X_i\}_{i \in \omega}$ are exactly the hyperarithmetical sets (X such that $X \leq_T 0^{(\alpha)}$ for some $\alpha < \omega_1^{\text{CK}}$), then U is Π_1^1 -complete (in particular not Σ_α^0 for any α). Kleene's $\mathcal{O}$ suffices to compute those sets.

Comparing the hierarchies: Subsets with high lightface complexity

There are sets that have genuinely high Borel ranks.

Example

$\{X \mid \omega_1^X = \omega_1^{\text{CK}}\}$ is properly $\mathbf{\Pi}_{\omega_1^{\text{CK}}+2}^0$, so it cannot be Π_α^0 for any $\alpha < \omega_1^{\text{CK}}$. In fact, it is Σ_1^1 -complete.

Applications to computable structures: Computable structures

A mathematical structure can be a group/ring/graph/order/... For example, the natural numbers $(\mathbb{N}, +, \cdot, 0, 1, <)$ can be viewed as an ordered semiring. In general, to specify a structure we need the following:

- The set of its elements. (e.g. $\mathbb{N}$)
- The language/signature of available operations. (e.g. $\mathcal{L} = \{+, \cdot, 0, 1, <\}$)
- The atomic diagram, consisting of true quantifier-free statements. (e.g. $1 + 1 = 2, 0 \neq 1$)

For a structure to be computable, it needs to additionally:

- Have $\mathbb{N}$ as its set of elements;
- Have a computable language (possibly finite);
- Have a computable atomic diagram.

Applications to computable structures: Syntactic definitions

Work in a computable language $\mathcal{L}$, and fix an enumeration $\{\varphi_i\}_{i \in \omega}$ of atomic $\mathcal{L} \cup \mathbb{N}$ -sentences (viewing $\mathbb{N}$ as constants). For any $\mathcal{L}$ -structure $\mathcal{M}$, its atomic diagram can be encoded by the path $p_{\mathcal{M}} \in 2^{\mathbb{N}}$, where $p_{\mathcal{M}}(i) = 1 \iff \mathcal{M} \models \varphi_i$ and 0 otherwise. Identify $\mathcal{M}$ with $p_{\mathcal{M}}$, noting that $\mathcal{M}$ is computable iff so is $p_{\mathcal{M}}$.

Given a formula φ , let $\text{Mod}(\varphi) \subseteq 2^{\mathbb{N}}$ be the set of all structures that satisfy φ . If φ is quantifier-free then $\text{Mod}(\varphi)$ is clopen, and quantifiers correspond to countable unions/intersections as usual. Hence, $\text{Mod}(\varphi)$ is always Borel. However, one cannot always recover first-order formulas from a Borel subset, so we need a larger class of formulas.

Applications to computable structures: Syntactic definitions

We extend first-order logic to the infinitary logic $\mathcal{L}_{\omega_1\omega}$ by adding countable conjunctions and disjunctions, while quantification remains limited to finitely many variables. Countable conjunctions and disjunctions will be denoted by special symbols: $\bigwedge, \bigvee$.

The following hierarchy is analogous to the arithmetical hierarchy, and we treat countable conjunctions/disjunctions as universal/existential quantifiers.

Definition

- $\Sigma_0^{\text{in}} = \Pi_0^{\text{in}}$ consists of all *finitary* quantifier-free formulas.
- For a countable $\alpha > 0$ and formulas φ, ψ , say φ is $\Sigma_\alpha^{\text{in}}$ and ψ is Π_α^{in} if

$$\varphi \equiv \bigvee_{i \in I} \exists \bar{x} \chi_i(\bar{x}), \psi \equiv \bigwedge_{i \in I} \forall \bar{x} \theta_i(\bar{x}),$$

where each χ_i is Π_β^{in} and each θ_i is Σ_β^{in} for some $\beta < \alpha$.

Applications to computable structures: Syntactic definitions

If the conjunctions and disjunctions are c.e., we obtain the *computable infinitary* formulas; These are denoted by $\Sigma_\alpha^c, \Pi_\alpha^c$, and can be relativized to an oracle.

The following theorem gives the exact correspondence between infinitary formulas and Borel complexity of the set of models.

Theorem (López-Escobar; Vaught; Vanden Boom)

If $A \subseteq 2^\mathbb{N}$ is isomorphism-invariant, then:

- *A is Borel iff $A = \text{Mod}(\psi)$ for some infinitary sentence ψ .*
- *A is Π_α^0 iff $A = \text{Mod}(\psi)$ for some $\psi \in \Pi_\alpha^{\text{in}}$.*
- *A is Π_α^0 iff $A = \text{Mod}(\psi)$ for some $\psi \in \Pi_\alpha^c$. (This can be relativized to an oracle X .)*

Applications to computable structures: Syntactic definitions

Infinitary logic turns out to be strong enough to characterize countable structures up to isomorphism (unlike first-order logic).

Theorem (Scott Isomorphism Theorem)

Every countable structure $\mathcal{M}$ has an infinitary Scott sentence, i.e. a sentence φ which characterizes $\mathcal{M}$ up to isomorphism among countable structures.

The Scott sentence allows us to measure the complexity of a countable model, both lightface and boldface, by the previous theorem. Specifically, we can look at the “least” complicated Scott sentence (infinitary or computable infinitary).

Remark

For relativized formulas, a Π_α^{in} formula computable from $0^{(\beta)}$ can be expressed as a $\Pi_{\beta+\alpha}^c$ formula.

We now turn to the question of how closely the lightface and boldface complexities of a structure align. Roughly speaking, a structure is *friendly* when both complexities align. Examples include $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and many other natural structures.

When the structure is computable, the two complexities are not too far apart.

Theorem

Any computable structure with a Π_{α}^{in} Scott sentence has a $\Pi_{2\alpha}^c$ Scott sentence.

One might hope to improve the bound from 2α to α . However, it turns out that *unfriendly/belligerent* structures, even computable ones, exist at various levels, so such improvements are too good to be true.

Theorem (Harrison-Trainor; Andrews, Gonzalez, Z.)

For any computable ordinal $\alpha \geq 2$, there exists a computable structure with a Π_α^{in} Scott sentence but has no $\Sigma_{2\alpha}^c$ Scott sentence.

Thus the $\Pi_{2\alpha}^c$ bound for the previous theorem is tight.

Applications to computable structures: Unfriendliness/Belligerence

In addition, we can investigate the oracles that make belligerent structures (relatively) friendly; The number of jumps required is exactly the difference between 2α and α .

Definition

Given an ordinal α , $\gamma(\alpha)$ is the least γ such that $\gamma + \alpha \geq 2\alpha$. Specifically:

- For $\alpha = n \in \mathbb{N}$, $\gamma(n) = n$.
- For $\alpha = \lambda$ a limit ordinal, $\gamma(\lambda) = 0$.
- Otherwise, $\gamma(\alpha)$ is the greatest additively closed ordinal below α .
(Equivalently, the leading term in the Cantor normal form of α .)

Theorem (Harrison-Trainor; Andrews, Gonzalez, Z.)

Any computable structure with a Π_α^{in} Scott sentence has a Π_α^{in} Scott sentence computable from $0^{(\gamma(\alpha))}$.

The following theorem shows the optimality of $\gamma(\alpha)$.

Theorem (Harrison-Trainor; Andrews, Gonzalez, Z.)

For any computable ordinal $\alpha \geq 2$, there exists a computable structure such that:

- $\mathcal{A}$ has a Π_α^{in} Scott sentence;
- $\mathcal{A}$ has a $0^{(\gamma(\alpha))}$ -computable Π_α^{in} Scott sentence; and
- $\mathcal{A}$ has no $0^{(\beta)}$ -computable Π_α^{in} Scott sentence for any $\beta < \gamma(\alpha)$.

So the $\gamma(\alpha)$ bound for the previous theorem is also optimal.

Thank you for listening!

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